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A common fixed point result for two pairs of maps in b-metric spaces without (E.A.)-property

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ABSTRACT. In this paper, we investigate a common fixed point problem for two pairs $\{f, S\}$ and $\{g, T\}$ of weakly compatible selfmaps of a complete b-metric $(X, d; s)$, satisfying a contractive condition of Ćirić type. This contraction and some of its variants were used in the paper [29] published in 2016 by V. Ozturk and S. Radenovic, requiring the (E.A.)-property for the pairs $\{f, S\}$ and $\{g, T\}$. The aim of this paper is to provide some improvements to the main result of [29]. Our main theorem will improve certain results published in 2015, by V. Ozturk and D. Turkoglu (see [30] and [31]). We improve also results from other related papers (see the references herin). Indeed, we remove the (E.A.)-property and weaken certain assumptions imposed in these papers. So, our work aims to extend and unify, in one go, several common fixed point results known in a recent literature. We furnish two illustrative examples and we prove that the fixed point problem, considered here, for the pairs $\{f, S\}$ and $\{g, T\}$ is well-posed. We compare our main result with a recent result obtained in 2018 by N. Hussain, Z. D. Mitrović and S. Radenović in [19].

1. INTRODUCTION AND PRELIMINARIES

We start by recalling the following definition (see [8], [16]).

Definition 1.1. Let X be a nonempty set. A *b-metric* on X is a function $d : X \times X \to [0, \infty)$ satisfying the conditions

$$\begin{aligned} &\text{(i)} \quad d(x,y) = 0 \iff x = y, \\ &\text{(ii)} \quad d(x,y) = d(y,x), \\ &\text{(iii)} \quad d(x,y) \le s[d(x,z) + d(z,y)], \end{aligned} \tag{1}$$

for all $x, y, z \in X$, and for some fixed number $s \ge 1$.

The triple $(X, d; s)$ is called a b-*metric space* with parameter s.

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The inequality (iii) is called the s-relaxed triangle inequality or simply the s-triangle inequality.

Obviously, for $s = 1$ the function d becomes a metric on X. In this case the triple $(X, d; 1)$ is simply denoted by (X, d) which is the usual notation for a metric space X endowed with the metric d.

Let $(X, d; s)$ be a b-metric space with constant $s \geq 1$. The ball $B(x, r)$ of center $x \in X$ and radius $r > 0$ is defined by setting

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

A nonempty subset Y of X is called open if for every $x \in Y$ there exists a number $r_x > 0$ such that $B(x, r_x) \subset Y$. The empty set is open by definition.

We denote by $\mathcal{T}_d$ (or $\mathcal{T}(d)$) the family of all open subsets of X it follows that $\mathcal{T}_d$ satisfies the axioms of a topology. This topology $\mathcal{T}_d$ is metrizable (see for example [14] and the references therein). As a consequence of this, for every sequence (x_n) and for each $x \in X$, we have the following equivalence: (x_n) converges to x in the topological space $(X, \mathcal{T}_d)$ if, and only if, $\lim_{n\to+\infty} d(x_n, x) = 0$.

If $\lim_{n\to+\infty} d(x_n, x) = 0$, then we write $\lim_{n\to+\infty} x_n = x$.

Since the topological space $(X, \mathcal{T}_d)$ is Hausdorff the limit of a converging sequence is unique.

In general, the map $(x, y) \mapsto d(x, y)$ fails to be continuous on the topological product space $X \times X$ and the balls $B(x, r)$ are not be open sets (see for instance [14] and [32]).

A sequence (x_n) of points of X is said Cauchy sequence if and only if $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$.

It is easy to see that every converging sequence is a Cauchy sequence.

The following lemma (see [43] or [21]) is useful.

Lemma 1.1. [43] *Let $(X, d; s)$ be a b-metric space and $(y_n)_n$ a sequence in X such that*

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}), \quad n = 0, 1, 2, \ldots$$

where $0 \leq \lambda < 1$. Then the sequence $(y_n)_n$ is Cauchy sequence in X provided that $s\lambda < 1$.

The above lemma was improved in 2017 by R. Miculescu and A. Mihail in their paper [26] and by T. Suzuki in his paper [44] by a different method. Thus, we have the following fundamental lemma.

Lemma 1.2 ([26] and [44])**.** *Let $(X, d; s)$ be a b-metric space with parameter $s \geq 1$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements from X having the property that there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}), \quad \forall n \in \mathbb{N}.$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

In [18], T. M. Došenović, M. V. Pavlović and S. N. Radenović made a discussion concerning Lemma 1.2 above and showed that various known fixed point results in b-metric spaces can be shortened by the use of this fundamental lemma.

Lemma 1.2 was used by Nawab Hussain, Zoran D. Mitrović and Stojan Radenović in [19] to establish a Fisher contraction principle in b-metric space without continuity of the b-metric function.

The b-metric space (X, d, s) is said to be complete iff every Cauchy sequences converges in X.

A subset Y of X is said to be closed if its complementary set $Y^c := X \setminus Y$ is open (i.e., $Y^c \in \mathcal{T}_d$).

Let Y be a non empty set of the b-metric space $(X, d; s)$. We denote d_Y the restriction of d to the set $Y \times Y$. Then the space $(Y, d_Y; s)$ is b-metric space, called a b-metric subspace of X. We observe that if $(Y, d_Y; s)$ is a complete b-metric space, then Y is closed in $(X, d; s)$.

For each subset Y of X, we denote $\overline{Y}$ the closure of Y. That is the smallest closed subset of X containing Y. It is easy to prove the following two lemmas.

Lemma 1.3. *Let $(X, d; s)$ be a b-metric space and let Y be a subset of X. Let $x \in X$. Then the following assertions are equivalent:*

(a) $x \in \overline{Y}$.

(b) *There exists a sequence $(y_n)_n$ of points in Y which converges to x (i.e., $\lim_{n\to+\infty} d(x, y_n) = 0$).*

(c) $d(x, Y) = 0$, *where* $d(x, Y) := \inf\{d(x, y) : y \in Y\}$.

Lemma 1.4. *Let $(X, d; s)$ be a b-metric space and let Y be a subset of X. Then the following assertions are equivalent:*

(a) *Y is closed.*

(b) $Y = \overline{Y}$.

(c) *For each sequence $(y_n)_n$ of points in Y and for every $x \in X$, if $\lim_{n\to+\infty} d(x, y_n) = 0$, then we have $x \in Y$.*

The property (E.A.) introduced in 2002 by Aamri and Moutawakil [1] for metric spaces can also be extended to b-metric spaces as follows.

Definition 1.2. Let f and g be two selfmappings of a b-metric space $(X, d; s)$. We say that f and g satisfy property (E.A.) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

The concept of compatible selfmaps of metric spaces was first introduced by Jungck in [22]. As in [19], one can extend this concept to the context of b-metric spaces as follows.

Definition 1.3. Two selfmappings S and T of a b-metric space $(X, d; s)$ are called compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some t in X.

To precise more terminology, we recall the following definition.

Definition 1.4. Let X be a nonempty set and f, g selfmappings of X.

A point $u \in X$ is called a coincidence point of f and g if $fu = gu$.

A point $z \in X$ is called a point of coincidence of f and g if there exists a point $u \in X$ of f and g such that $z = fu = gu$.

Let f and g be selfmappings of X.

The set of coincidence points of f and g is denoted by $\operatorname{Coin}(f, g)$ and is given by $\operatorname{Coin}(f, g) := \{u \in X : f(u) = g(u)\}$.

The set of points of coincidence of f and g is denoted by $\operatorname{Poc}(f, g)$ and is given by $\operatorname{Poc}(f, g) := \{y \in X : \exists u \in X, \text{ such that } y = f(u) = g(u)\}$.

It is clear that $\operatorname{Poc}(f, g) = f(\operatorname{Coin}(f, g)) = g(\operatorname{Coin}(f, g))$.

The following definition was introduced (in 1996) by Jungck (see [23]) in the setting of metric fixed point theory.

Definition 1.5. ([23]) Two selfmaps f and g of a nonempty set X are called weakly compatible maps if they commute at every coincidence point of f and g (i.e., for all $u \in X$, $fu = gu \Longrightarrow fg(u) = gf(u)$).

It is easy to see that compatible selfmaps of a b-metric space are weakly and that the converse is not true.

During all this paper, $(X, d; s)$ will be a complete b-metric space with constant $s \geq 1$. For any given maps $f, g, S, T : X \to X$, we define the following functions on $X \times X$ by setting

(2)
$$M_s(x, y) := \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx, gy) + d(Ty, fx)}{2s}\right\}$$

and

(3)
$$N_s(x, y) := \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx, gy)}{2s}, \frac{d(Ty, fx)}{2s}\right\}.$$

We observe that $N_s(x, y) \leq M_s(x, y)$, for all $x, y \in X$.

Let Φ be the set of the continuous functions $\varphi : [0, +\infty) \to [0, +\infty)$ which are nondecreasing and such that $\varphi(t) = 0 \Longleftrightarrow t = 0$.

The following common fixed point result was established in [30].

Theorem 1.1. *(Theorem 2.1 in* [30]*) Let $(X, d; s)$ be a complete b-metric space with constant $s \geq 1$ and let $f, g, S, T : X \to X$ be mappings such that $f(X) \subseteq TX$ and $g(X) \subseteq SX$. Suppose that there exist $\psi, \varphi \in \Phi$ such that, such that for all $x, y \in X$, we have*

$$\psi\left(s^2 d(fx, fy)\right) \leq \psi\left(M_s(x, y)\right) - \phi\left(M_s(x, y)\right). \tag{4}$$

Suppose that one of the pairs $\{f, S\}$ and $\{g, T\}$ satisfy the (E.A.)-property and that one of the subspaces $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is closed in X. Then the pairs $\{f, S\}$ and $\{g, T\}$ have a point of coincidence in X.

Moreover, if the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

To improve Theorem 1.1 above together with another result established by Ozturk and Turkoglu in [31], Ozturk and Radenovic (see [29]) have proved the following theorem.

Theorem 1.2. *([29]) Let $(X, d; s)$ be a complete b-metric space with constant $s > 1$ and let $f, g, S, T : X \to X$ be mappings with $f(X) \subseteq TX$ and $g(X) \subseteq SX$ such that*

(5)

$$s^{\varepsilon} d(fx, gy) \leq \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx, gy) + d(Ty, fx)}{2s}\right\}$$

for all $x, y \in X$ and $\varepsilon > 1$ is a constant. Suppose that one of the pairs $\{f, S\}$ and $\{g, T\}$ satisfy the property (E.A.) and that one of the subspaces $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is closed in X. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Under the assumptions of Theorem above, the condition (5) is equivalent to the following:

(6)

$$d(fx, gy) \leq \alpha(s) \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx, gy) + d(Ty, fx)}{2s}\right\},$$

for all $x, y \in X$, where $\alpha(s) := \frac{1}{s^{\varepsilon}}$. We observe here that $s\alpha(s) = s^{1-\varepsilon} < 1$.

A similar result (see Theorem 15 of) [29] was furnished in [29] for selfmaps fulfilling the following contractive condition:

(7)

$$d(fx, gy) \leq \alpha(s) \max\left\{d(Sx, Ty), \frac{d(Sx, fx) + d(Ty, gy)}{2s}, \frac{d(Sx, gy) + d(Ty, fx)}{2s}\right\},$$

for all $x, y \in X$.

Then it is clear that Theorem 15 of [29] is a simple consequence of Theorem 1.2.

We observe also that the condition (4) implies the condition (5) for $\varepsilon = 2$ and that Theorem 1.2 is not valid for metric spaces, since $s > 1$.

In [21], b-metric spaces were called metric type spaces and the following result was established.

Theorem 1.3. *(Theorem 3.11 in [21]) Let $(X, d; s)$ be a complete b-metric space with constant $s \geq 1$, and and let $f, g : X \to X$ be two mappings such that $f(X) \subseteq g(X)$ and one of these subsets of X is complete. Suppose that* *there exists* $\lambda \in (0, \frac{1}{s})$ *such that for all* $x, y \in X$

(8)

$$d(fx, fy) \leq \lambda \max\left\{d(gx, gy), d(fx, gx), d(fy, gy), \frac{d(fx, gy)}{2s}, \frac{d(gy, fx)}{2s}\right\}.$$

Then f *and* g *have a unique point of coincidence. If, moreover, the pair* $\{f, g\}$ *is weakly compatible, then* f *and* g *have a unique common fixed point.*

The following common fixed point result for four continuous mappings on b-metric spaces was established in [19].

Theorem 1.4 (Theorem 2.1 in [19])**.** *Let* $(X, d; s)$ *be a complete b-metric space with constant* $s \geq 1$*, and and let the pairs* (S, I) *and* (T, J) *be b-compatible pairs of selfmaps of* X *satisfying*

(9)

$$d(Sx, Ty) \leq \lambda \max\left\{d(Ix, Jy), d(Sx, Ix), d(Ty, Jy), \frac{d(Sx, Jy) + d(Ty, Ix)}{2s}\right\},$$

for all $x, y \in X$*, where* $0 \leq \lambda < 1$*. If* $S(X) \subseteq J(X)$*,* $T(X) \subseteq I(X)$ *and if* I, J, S *and* T *are continuous, then* I, J, S *and* T *have a unique common fixed point.*

In this paper, we continue the discussion started in [29] conerning the results of [31].

One of the aims of this paper is to improve the results of [31] and [29] without using the b-(E.A.)-property. In particular, the first main result of this paper provides an improvement to Theorem 1.2. Indeed, we establish in Theorem 2.1 that one can remove the assumption (made in Theorem 1.2) requiring that one of the pairs $\{f, S\}$ and $\{g, T\}$ satisfies the b-(E.A.)-property. Besides, the condition requiring the closedness (in X) of one of the subspaces $f(X)$, $g(X)$, $S(X)$ and $T(X)$, in the b-metric space X, will be relaxed. Also, the particular constant $\alpha(s) := \frac{1}{s^{\varepsilon}}$ will be replaced by any other constant λ (which may depend or not on the parameter s) satisfying only the condition $s\lambda < 1$. Also, our first main result (see Theorem 2.1) will extend and complete Theorem 1.3 proved by M. Jovanović, Z. Kadelburg and S. Radenović in [21].

Our main result provides a general common fixed point result extending and unifying the results stated in Theorem 1.1, Theorem 1.2 and Theorem 1.3 together with other related results.

The contractive condition (C.3) used in Theorem 2.1 below, for the case of b-metric spaces, may be seen as the analogous of the Ćirić's (generalized) contractive condition, investigated by Lj. B. Ćirić in [13] in context of metric spaces.

This paper is organized as follows:

After this introduction, in section 2, we establish our first main result (see Theorem 2.1). We end this section by providing an illustrative example.

In section 3, we dispaly a list of related results which are easy consequences of Theorem 2.1 and discuss the relationship with this theorem and the above results recalled in the introduction.

In section 4, we establish the well-posedness of the fixed point problem studied in Theorem 2.1.

2. Main result

The first main result of this paper reads as follows.

Theorem 2.1. *Let $(X, d; s)$ be a complete b-metric space with constant $s \geq 1$. Let $\{f, S\}$ and $\{g, T\}$ be two pairs of selfmappings of X satisfying the following conditions:*

(C.1) : $\overline{f(X)} \subseteq \overline{T(X)}$ *and* $g(X) \subseteq S(X)$,

(C.2) : $\overline{f(X)} \cap \overline{g(X)} \subset T(X) \cup S(X)$.

(C.3) : $d(fx, gy) \leq \lambda M_s(x, y)$, *for all* $x, y \in X$, *where* $\lambda \in [0, \frac{1}{s})$ *and* $M_s(x, y) := \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx,gy)+d(Ty,fx)}{2s}\right\}$, *for all* $x, y \in X$.

Then the pairs (f, S) and (g, T) have a unique common point of coincidence in X. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. (I) Let x_0 be any point in X. Define sequences (x_n) and (y_n) in X as follows:

$$y_{2n} := fx_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} := gx_{2n+1} = Sx_{2n+2}, \ \forall n \geq 0.$$

The existence of such sequences is ensured by the condition (C.1).

(II) Next, we prove that the sequence $\{y_n\}$ is a Cauchy sequence in X.

To simplify notations, we set $\tau_n := d(y_n, y_{n+1})$ for all non-negative integer n.

By using the assumption (C.3) and the s-triangle inequality, for all $n \geq 0$, we have

$$\begin{aligned}
& d(y_{2n}, y_{2n+1}) = d(fx_{2n}, gx_{2n+1}) \\
\leq\ & \lambda \max\Big\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1}), \\
& \qquad \frac{d(Sx_{2n}, gx_{2n+1}) + d(Tx_{2n+1}, fx_{2n})}{2s}\Big\} \\
=\ & \lambda \max\Big\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n+1})}{2s}\Big\} \\
\leq\ & \lambda \max\Big\{d(y_{2n-1}, y_{2n}), , d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{2}\Big\} \\
=\ & \lambda \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}.
\end{aligned}$$

Therefore, we have $\tau_{2n} \leq \lambda \max\{\tau_{2n-1}, \tau_{2n}\}$, for all non-negative integer n.

Suppose that $\tau_{2n} > \tau_{2n-1}$ for some intger $n \geq 0$. Then we would have $\tau_{2n} > 0$ and $(1-\lambda)\tau_{2n} \leq 0$, which implies that $\tau_{2n} = 0$, because $1-\lambda > 0$. This is a contradiction. Thus we have showed that

$$\tau_{2n} \leq \lambda\, \tau_{2n-1}, \quad \text{for all integer}\, n \geq 1. \tag{10}$$

By similar arguments, we prove that

$$\tau_{2n+1} \leq \lambda \tau_{2n}, \quad \text{for all integer}\, n \geq 0. \tag{11}$$

From the inequalities (10) and (11), we deduce that

$$\tau_n \leq \lambda\, \tau_{n-1}, \quad \text{for all integer}\, n \geq 1, \tag{12}$$

Since by assumption, we have $0 \leq s\lambda < 1$, then $0 \leq \lambda < \frac{1}{s} \leq 1$. Thus we can use Lemma 1.1 (or Lemma 1.2) and conclude that the sequence $\{y_n\}$ is a Cauchy sequence in the b-metric space $(X, d; s)$. Since the b-metric space $(X, d; s)$ is complete, there exists some point $z \in X$ such that $z = \lim_{n\to+\infty} y_n$. Therefore, we have

$$z = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n}. \tag{13}$$

(III) Next, we show that z is the unique common point of coincidence in X for the pairs (f, S) and (g, T). Indeed, by virtue of (C.2) and the equalities (13), we deduce that $z \in T(X) \cup S(X)$. Thus, there are two cases to be discussed:

(i) Suppose that $z \in T(X)$, then there exists $v \in X$ such that $z = Tv$. By applying the s-triangle inequality, we obtain

$$\begin{aligned} \frac{1}{s} d(gv, z) &\leq d(gv, fx_{2n}) + d(fx_{2n}, z) \\ &= d(gv, y_{2n}) + d(y_{2n}, z). \end{aligned}$$

By taking the limits in the above inequalities, we obtain

$$\frac{1}{s} d(gv, z) \leq \liminf_{n\to+\infty} d(gv, y_{2n}). \tag{14}$$

Besides, by applying (C.3), we have

$$\begin{aligned} &d(gv, y_{2n}) = d(fx_{2n}, gv) \\ &\leq \lambda \max\Big\{d(Sx_{2n}, Tv), d(Sx_{2n}, fx_{2n}), d(Tv, gv), \frac{d(Sx_{2n}, gv) + d(Tv, fx_{2n})}{2s}\Big\} \\ &= \lambda \max\Big\{d(y_{2n-1}, z), \tau_{2n-1}, d(gv, z), \frac{d(y_{2n-1}, gv) + d(z, y_{2n})}{2s}\Big\} \\ &\leq \lambda \max\Big\{d(y_{2n-1}, z), \tau_{2n-1}, d(gv, z), \frac{sd(y_{2n-1}, z) + sd(z, gv) + d(z, y_{2n})}{2s}\Big\} \\ &= \lambda \max\Big\{d(y_{2n-1}, z), \tau_{2n-1}, d(gv, z), \frac{d(y_{2n-1}, z)}{2} + \frac{d(z, gv)}{2} + \frac{d(z, y_{2n})}{2s}\Big\}. \end{aligned}$$

From which, we deduce that, for all positive integer n, we have

$$d(gv, y_{2n}) \leq \lambda \max\Big\{d(gv, z) + \tau_{2n-1} + d(y_{2n-1}, z) + d(z, y_{2n})\Big\}, \tag{15}$$

By taking the limits in both sides of (15), we get

$$\limsup_{n\to+\infty} d(gv, y_{2n}) \leq \lambda\, d(gv, z). \tag{16}$$

From inequalities (14) and (16), we infer that $d(gv, z) \leq s\lambda d(gv, z)$, which implies that $(1 - s\lambda)d(gv, z) \leq 0$. Since $1 - s\lambda > 0$, then we conclude that $z = gv$. Therefore we obtain

$$z = gv = Tv. \tag{17}$$

Hence, z is a point of coincidence of the pair $\{g, T\}$.

(ii) Suppose that $z \in S(X)$, then by similar arguments, one will ensure the existence of a point $u \in X$ such that

$$z = fu = Su. \tag{18}$$

Thus z is common point of coincidence for both pairs $\{f, S\}$ and $\{g, T\}$. That is $z \in \mathrm{Poc}\,(f, S) \cap \mathrm{Poc}\,(g, T)$.

Let $w \in \mathrm{Poc}\,(f, S)$ and let $a \in X$ be such that $w = f(a) = g(a)$. Then according to (C.3) and a short computation will show that we have

$$d(w, z) = d(fa, gv) \leq \lambda\, M_s(w, z) = \lambda\, d(w, z),$$

which implies that $d(w, z) = 0$, that is $w = z$. Therefore, we have $\mathrm{Poc}\,(f, S) = \{z\}$.

By a similar way, we get $\mathrm{Poc}\,(g, T) = \{z\}$.

We conclude that we have proved the following equalities:

$$\mathrm{Poc}\,(f, S) = \mathrm{Poc}\,(g, T) = \{z\} = \mathrm{Poc}\,(f, S) \cap \mathrm{Poc}\,(g, T). \tag{19}$$

(19) says that z is the unique point of coincidence of each one of the pairs $\{f, S\}$ and $\{g, T\}$ and that z is the unique common point of coincidence of both pairs $\{f, S\}$ and $\{g, T\}$.

(IV) Suppose now, that the pairs (f, S) and (g, T) are weakly compatible. Then from (17) and (18), we deduce that

$$gz = Tz \quad \text{and} \quad fz = Sz. \tag{20}$$

(20) shows that $gz \in \mathrm{Poc}\,(g, T) = \{z\}$ and that $fz \in \mathrm{Poc}\,(f, S) = \{z\}$. Therefore, we get $z = gz = Tz$ and $z = fz = Sz$.

We conclude that z is a common fixed point of the selfmappings f, g, S and T.

The uniqueness of z is ensured by the equalities (19). This ends the proof. □

We end this section by giving an illustrative example.

Example 2.1. Let $X := [0, +\infty)$ be equipped with the b-metric d given by $d(x, y) := |x - y|^2$ for all $x, y \in X$. The parameter of the b-metric d is $s = 2$. We define four maps f, g, S and T on X by setting

$$f(x) := \ln(1 + \tfrac{x}{2}), \qquad g(x) := \ln(1 + \tfrac{x}{3}),$$

$$S(x) := e^{3x} - 1, \qquad T(x) := e^{2x} - 1.$$

Then we have:

(1) The b-metric space $(X, d; 2)$ is complete.

(2) $AX = BX = SX = TX = X$.

(3) From (2) and (1), we deduce that the conditions (C.1) and (C.2) are satisfied.

(4) The pair $\{f, S\}$ is weakly compatible. Indeed, for all $x \in X$, we have

$$fx = Sx \Longleftrightarrow \ln(1 + \tfrac{x}{2}) = e^{3x} - 1 \Longleftrightarrow x = 0,$$

In that case, we have $fS(0) = Sf(0) = 0$.

(5) Similarly, we show that the pair $\{g, T\}$ is weakly compatible.

(6) For all $x, y \in X$, we have

$$\begin{aligned} d(fx, gy) &= |fx - gy|^2 = \left|\ln(1 + \tfrac{x}{2}) - \ln(1 + \tfrac{y}{3})\right|^2 \\ &\leq \left|\frac{x}{2} - \frac{y}{3}\right|^2 = \frac{1}{36}|3x - 2y|^2 \\ &\leq \frac{1}{36}\left|e^{3x} - e^{2y}\right|^2 = \frac{1}{36} d(Sx, Ty) \\ &\leq \frac{1}{36} \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx, gy) + d(Ty, fx)}{4}\right\}. \end{aligned}$$

We set $\lambda = \frac{1}{36}$. We observe, here, that $s\lambda = 2\lambda = \frac{1}{18} < 1$.

According to (6), we see that the assumption (C.3) is satisfied.

We conclude that all conditions of Theorem 2.1 are satisfied and the unique common fixed point of the selfmappings f, g, S and T is zero.

3. Consequences and related results

3.1. **Consequences.** In this subsection, we dispaly a list of common fixed point results which are direct or easy consequences of Theorem 2.1.

We start with a variant of Theorem 2.1.

Theorem 3.1. *Let $(X, d; s)$ be a complete b-metric space with constant $s \geq 1$. Let $\{f, S\}$ and $\{g, T\}$ be two pairs of selfmappings of X satisfying the following conditions:*

(C.1) : *$f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$.*

(C.2)$'$: *One of the sets $f(X), g(X), T(X)$ or $S(X)$ is closed in X.*

(C.3) : $d(fx, gy) \leq \lambda \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx,gy)+d(Ty,fx)}{2s}\right\}$,

for all $x, y \in X$, where λ is a constant such that $0 < s\lambda < 1$.

Then the pairs (f,S) and (g,T) have a unique common point of coincidence in X. Moreover, if the pairs (f,S) and (g,T) are weakly compatible, then f,g,S and T have a unique common fixed point.

Indeed, the conditions (C.1) and (C.2)$'$ imply (C.2).

The next result is another variant of Theorem 2.1.

Theorem 3.2. *Let $(X,d;s)$ be a complete b-metric space with constant $s \geq 1$. Let $\{f,S\}$ and $\{g,T\}$ be two pairs of selfmappings of X satisfying the following conditions:*

(i) $\overline{f(X)} \subseteq T(X)$ *and* $\overline{g(X)} \subseteq S(X)$,

(ii) $d(fx,gy) \leq \lambda \max\left\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{d(Sx,gy)+d(Ty,fx)}{2s}\right\}$,

for all $x,y \in X$, where λ is a constant satisfying: $0 < s\lambda < 1$.

Then the pairs (f,S) and (g,T) have a unique common point of coincidence in X. Moreover, if the pairs (f,S) and (g,T) are weakly compatible, then f,g,S and T have a unique common fixed point.

In the next result, we provide an improvement of Theorem 1.3.

Theorem 3.3. *Let $(X,d;s)$ be a complete b-metric space with constant $s \geq 1$, and and let $f,g,S,T : X \to X$ be two mappings such that*

(a) $f(X) \subset S(X)$ and one of these subsets of X is complete.

(b) $g(X) \subset T(X)$ and one of these subsets of X is complete.

Suppose that there exists $\lambda \in (0,\frac{1}{s})$ such that for all $x,y \in X$

(21)
$$d(fx,gy) \leq \lambda \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty)}{2s}, \frac{d(gy,fx)}{2s}\right\},$$

Then the pairs $\{f,S\}$ and $\{g,T\}$ have a unique point of coincidence. If, moreover, the pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then fng,S and T have a unique common fixed point.

This extends a result exposed in Theorem 1.3 (see Theorem 3.11 in [21]). Indeed, the conditions (a) and (b) infer that $\overline{f(X)} \subset S(X)$ and $\overline{g(X)} \subset T(X)$. We recall that $N_s \leq M_s$, so we can use Theorem 3.2 and recapture the above result.

3.2. **Relationship with Theorem 1.4.** We end this section by a comment on the relationship between our Theorem 2.1 and the main result (Theorem 1.4) of [19]. It is worthy to notice that these two results are of different nature. Next, we furnish an example of a situation where our result can be applied but not Theorem 2.1 of [19].

Example. Let $X := [0,1]$ be endowed with the b-metric $d(x,y) := (x-y)^2$ for all $x,y \in X$. We consider functions $S,T : X \to X$ defined by

$$Sx = Tx := \begin{cases} x, & \text{if } x \in [0,\frac{1}{2}]; \\ 1, & \text{if } x \in]\frac{1}{2},1]. \end{cases}$$

We consider functions $f, g : X \to X$ defined by

$$fx = gx := \begin{cases} \frac{x}{2}, & \text{if } x \in [0, \frac{1}{2}]; \\ \frac{1}{2}, & \text{if } x \in]\frac{1}{2}, 1]. \end{cases}$$

Then we have the following properties:

(C 0) (X, d) is a complete b-metric space with parameter $s = 2$.

(C 1) $f(X) = g(X) = [0, \frac{1}{4}] \cup \{\frac{1}{2}\} \subseteq TX = SX = [0, \frac{1}{2}] \cup \{1\}$.

(C 2) $\overline{f(X)} \cap \overline{g(X)} = [0, \frac{1}{4}] \cup \{\frac{1}{2}\} \subseteq TX \cup SX = [0, \frac{1}{2}] \cup \{1\}$.

(C 3) For all $x, y \in X$, we have

$$\begin{aligned} d(fx, gy) = (fx - gy)^2 &= \left(\frac{Sx}{2} - \frac{Tx}{2}\right)^2 \\ &= \frac{1}{4}(Sx - Tx)^2 = \frac{1}{4} d(Sx, Ty) \\ &\leq \frac{1}{4} \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx, gy) + d(Ty, fx)}{4}\right\}. \end{aligned}$$

We set $\lambda = \frac{1}{4}$. We observe, here, that $s\lambda = 2\lambda = \frac{1}{2} < 1$.

Therefore, the assumption (C.3) is satisfied.

(C 4) The pairs (f, S) and (g, T) are weakly compatible. Indeed, for all $x \in X$, we have

$$fx = Sx \iff \frac{Sx}{2} = Sx \iff Sx = 0 \iff x = 0,$$

In that case, we have $fS(0) = Sf(0) = 0$.

Thus, all conditions of our Theorem 2.1 are satisfied. By applying this theorem we infer that the selfmaps f, g, S and T have a unique common fixed point (which is the point 0).

All the functions f, g, S and T are discontinuous at the the point $\frac{1}{2}$, so Theorem 1.4 which is the main result of [19] can not be applied.

4. Well-posedness

After the works of F. S. De Blasi and J. Myjak [10] and of S. Reich and A. J. Zaslavski [40], many authors have investigated the well-posedness of fixed point problems (see [25], [38], [42], [37], [39], [2], [3] and [6]).

The following definition was introduced in the setting of metric spaces.

Definition 4.1. Let (X, d) be a metric space and $T : (X, d) \to (X, d)$ be a mapping. The fixed point problem of T is said to be well posed if:

(i) T has a unique fixed point z in X,

(ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, we have $\lim_{n\to\infty} d(x_n, z) = 0$.

The above definition may be naturally extended to the context of b-metric spaces by the following definition (see for example [6]).

Definition 4.2. Let $(X, d; s)$ be a b-metric space with constant $s \geq 1$. Let $\mathcal{A}$ be a set of selfmappings $T : X \to X$. The fixed point problem of the collection $\mathcal{A}$ is said to be well-posed if:

(i) the set $\mathcal{A}$ has a unique strict fixed point z in X,

(ii) for any sequence $\{x_n\}$ of points in X such that

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0, \quad \forall T \in \mathcal{A},$$

we have $\lim_{n\to\infty} d(x_n, z) = 0$.

According to this definition, we investigate the well-posedness of the common fixed point problem for the set of four selfmappings f, g, S, T of a b-metric space $(X, d; s)$ satisfying the conditions of Theorem 2.1.

Theorem 4.1. *Let $\{f, S\}$ and $\{g, T\}$ be two weakly compatible pairs of selfmappings of a complete b-metric space $(X, d; s)$ such that*

$(C.1)$: $fX \subseteq TX$ *and* $gX \subseteq SX$,

$(C.2)$: $\overline{fX} \cap \overline{gX} \subset T(X) \cup S(X)$.

$(C.3)$: $d(fx, gy) \leq \lambda \max\left\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{d(Sx,gy)+d(Ty,fx)}{2s}\right\}$,

for all $x, y \in X$, where λ is such that $0 \leq s\lambda < 1$.

Then the fixed point problem of f, g, S and T is well-posed.

Proof. By Theorem 2.1, we know that the mappings f, g, S and T have a unique common fixed point z in X. Let $\{u_n\}$ be a sequence in X such that

(22)
$$\lim_{n\to\infty} d(fu_n, u_n) = \lim_{n\to\infty} d(Su_n, u_n) = \lim_{n\to\infty} d(gu_n, u_n) = \lim_{n\to\infty} d(Tu_n, u_n) = 0.$$

We want to show that $\lim_{n\to\infty} d(u_n, z) = 0$.

We start by observing that

$$d(fu_n, Su_n) \leq s[d(fu_n, u_n) + d(u_n, Su_n)],$$

which implies that $\lim_{n\to\infty} d(fu_n, Su_n) = 0$.

Now, by using the inequality (C.3) and the s-triangle inequality, we have successively

$$\begin{aligned}
&d(fu_n, z) = d(fu_n, gz) \\
\leq\ & \lambda \max\left\{d(Su_n, z), d(Su_n, fu_n), 0, \frac{1}{2s}[d(Su_n, z) + d(z, fu_n)]\right\} \\
\leq\ & \lambda \max\Big\{s(d(Su_n, fu_n) + d(fu_n, z)), d(Su_n, fu_n), \\
& \qquad \frac{1}{2s}[s(d(Su_n, fu_n) + d(fu_n, z)) + d(z, fu_n)]\Big\} \\
=\ & \lambda \max\left\{sd(fu_n, z) + sd(Su_n, fu_n), \frac{s+1}{2s} d(z, fu_n) + \frac{1}{2} d(Su_n, fu_n)\right\} \\
=\ & s\lambda\, [d(fu_n, z) + d(Su_n, fu_n)],
\end{aligned}$$

from which we obtain

$$d(fu_n, z) \leq \frac{s\lambda}{1 - s\lambda} d(Su_n, fu_n), \quad \forall n \geq 0. \tag{23}$$

The inequality (23) holds true because $1 - s\lambda > 0$.

Letting n go to infinity in (23), we get

$$\lim_{n\to\infty} d(fu_n, z) = 0. \tag{24}$$

By using the s-triagle inequality, for all nonnegative integer n, we have

$$d(u_n, z) \leq s\big(d(u_n, fu_n) + d(fu_n, z)\big),$$

from which (according to (22) and (24)), by letting n tend to infinity, we obtain that $\lim_{n\to\infty} d(u_n, z) = 0$. Henceforth, the fixed point problem for the mappings f, g, S and T is well posed. This ends the proof. □

We point out that Theorem 4.1 improves Theorem 18 of [29].

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